Graded q-pseudo-differential operators and supersymmetric algebras

2002 J. Phys. A: Math. Gen. 35 3697

(http://iopscience.iop.org/0305-4470/35/16/308)

J. Phys. A: Math. Gen. **35** (2002) 3697–3702 PII: S0305-4470(02)32930-5

Graded q-pseudo-differential operators and supersymmetric algebras

Ahmed Jellal[1] **and El Hassan El Kinani**[2,3]

[1] High Energy Physics Section, The Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, 34100 Trieste, Italy
[2]Mathematical Physics Group, Department of Mathematics, Faculty of Sciences and Technics, University Molay Ismail, PO Box 509, Errachidia, Morocco
[3] LPT, Department of Physics, Faculty of Sciences, University Med V, Rabat, Morocco

E-mail: jellal@ictp.trieste.it, jellal@youpy.co.uk, hkinani@ictp.trieste.it and elkinani@fste.ac.ma

Received 18 January 2002, in final form 15 March 2002
Published 12 April 2002
Online at stacks.iop.org/JPhysA/35/3697

Abstract
We give a supersymmetric generalization of the sine algebra and the quantum algebra $U_t(sl(2))$. Making use of the q-pseudo-differential operators graded with a fermionic algebra, we obtain a supersymmetric extension of sine algebra. With this scheme we also get a quantum superalgebra $U_t(sl(2/1))$.

PACS numbers: 02.20.−a, 03.65.−w, 11.30.Pb

1. Introduction

One of the most important infinite-dimensional Lie algebras is the one generated by the so-called pseudo-differential operators [1]. This can be viewed as a generalization of the Virasoro algebra and of the Lie algebra of differential operators. Recently, the supersymmetric algebras have been applied to the study of some physical problems. For example, the supersymmetric sine algebra is used to investigate the properties of Bloch electrons in a constant uniform magnetic field [2]. Moreover, the quantum superalgebras can be applied to solving some problems, for instance, related to superconductivity [3] and the quantum Hall effect [4]. These results led us to the present work.

In this paper, we will present an approach to obtain a realization of certain supersymmetric algebras. More precisely, we will propose a graded q-pseudo-differential operator realization of the supersymmetric extension of the sine algebra and the quantum superalgebra $U_t(sl(2/1))$.

This paper is organized as follows. In section 2 we review some basic notions related to q-pseudo-differential operators and also to the realization of sine algebra and $U_t(sl(2))$ in this framework. We propose a realization of the supersymmetric extension of the last algebras in section 3. We give our conclusion in the final section.

2. Prelimenaries

Before proceeding, we would like to give a short review concerning some basic notions, which will be useful in the next section. This concerns the q-pseudo-differential operators, sine algebra and $U_t(sl(2))$.

2.1. q-pseudo-differential operators

We start first by defining the so-called q-derivation. For this, let q be a complex number different from 0 and 1. By definition, the q-derivation or, more generally, the α-derivation is given by

$$d_\alpha(fg) = \alpha(f)d_\alpha(g) + d_\alpha(f)g, \tag{1}$$

where $f, g \in C[x, x^{-1}]$ are the ring of polynomials in an indeterminant x and its inverse x^{-1}. In equation (1), α is a linear mapping. An example of α-derivation is given by Jackson's q-differential operator ∂_q, such as [5]

$$\partial_q(f) = \frac{f(qx) - f(x)}{(q-1)x}, \tag{2}$$

which leads to the following form for equation (1):

$$\partial_q(fg) = \eta_q(f)\partial_q(g) + \partial_q(f)g, \tag{3}$$

where the shift operator η_q is

$$\eta_q(f(x)) = f(qx). \tag{4}$$

Now let us introduce the q-pseudo-differential operator algebra $q - \psi DO$. The latter is characterized by the relation [5]

$$q - \psi DO = P(x, \partial_q) = \sum_{i=-\infty}^{N} P_i(x)\partial_q^i, \qquad P_i(x) \in C[x, x^{-1}]. \tag{5}$$

Consequently, the algebra $q - \psi DO$ is generated by x, x^{-1}, ∂_q, ∂_q^{-1} with the relation

$$\partial_q x - qx\partial_q = 1. \tag{6}$$

Note that the family $\{x^i\partial_q^j\}_{i,j\in Z}$ forms a basis of $q - \psi DO$. Then, the algebra $q - \psi DO$ can be viewed as a Lie algebra, which can be defined by the commutation relation

$$[P, Q] = P \circ Q - Q \circ P, \tag{7}$$

where the multiplication law '$\circ$' is

$$\begin{aligned} &\partial_q \circ f = \eta_q(f)\partial_q + \partial_q f, \\ &\partial_q^{-1} \circ f = \sum_{k\geqslant 0}(-1)^k q^{-k(k+1)/2}(\eta_q^{-k-1}(\partial_q^k f))\partial_q^{-k-1}. \end{aligned} \tag{8}$$

The last equation is obtained by using the following relation:

$$\partial_q^{-1} \circ (\partial_q \circ f) = \partial_q \circ (\partial_q^{-1} \circ f) = f. \tag{9}$$

Note that equation (8) can be unified as follows:

$$\partial_q^n \circ f = \sum_{k\geqslant 0} \binom{n}{k}_q (\eta_q^{n-k}(\partial_q^k f))\partial_q^{n-k}, \tag{10}$$

for all n. In the last equation, the q-binomials take the form

$$\binom{n}{k}_q = \frac{(n)_q(n-1)_q\cdots(n-k+1)_q}{(1)_q(2)_q\cdots(k)_q}, \tag{11}$$

and the q-numbers are given by

$$(n)_q = \frac{q^n-1}{q-1}, \tag{12}$$

where the convention

$$\binom{n}{0}_q = 1, \tag{13}$$

is taken. We also add that the residue of the symbol $P(x, \partial_q)$ can be written as

$$\mathrm{Res}\left(\sum_{i=-\infty}^{N} P_i(x)\partial_q^i\right) = P_{-1}(x) \tag{14}$$

and its Tr-functional is

$$\mathrm{Tr}\left(\sum_{i=-\infty}^{N} P_i(x)\partial_q^i\right) = \int_{S^1} P_{-1}(x)\,\mathrm{d}x. \tag{15}$$

The integral is taken on the circle $S^1 = \{\mathrm{e}^{\mathrm{i}\theta}, \theta \in \boldsymbol{R}\}$.

Considering a subfamily of $q - \psi DO$ as

$$q - S\psi DO = \left\{P(x, \partial_q) = \sum_{i=-\infty}^{N} P_i(x)(\partial_q)^i \,|\, \mathrm{Tr}(P) = 0\right\}. \tag{16}$$

From equation (4), we obtain the relation

$$\eta_q x = q x \eta_q. \tag{17}$$

Therefore, η_q and x generate a non-commutative algebra, which is homomorphic with Manin's plane or 'quantum plane' [6].

2.2. Sine algebra

In this subsection, we review the realization of the sine algebra and the quantum algebra $U_t(sl(2))$. To do this, let us introduce the following generators: $J_{\boldsymbol{m}}$, where $\boldsymbol{m} = (m_1, m_2)$ are vectors belonging to the square integer lattice $\boldsymbol{Z}^2 - \{(0,0)\}$. This can be constructed as follows [7]:

$$J_{\boldsymbol{m}} = q^{-m_1\cdot m_2/2}\eta_q^{m_1}x^{m_2}. \tag{18}$$

Calculating the commutation relation, it is found that

$$[J_{\boldsymbol{m}}, J_{\boldsymbol{n}}] = (q^{(\boldsymbol{m}\times\boldsymbol{n})/2} - q^{-(\boldsymbol{m}\times\boldsymbol{n})/2})J_{\boldsymbol{m}+\boldsymbol{n}}, \tag{19}$$

where $\boldsymbol{m}\times\boldsymbol{n} = m_1n_2 - m_2n_1$. It is interesting to note that, when q is a Fth root of unity, i.e. $q = \exp(\frac{4\pi\mathrm{i}}{F})$, the last equation takes the following form:

$$[J_{\boldsymbol{m}}, J_{\boldsymbol{n}}] = 2\mathrm{i}\sin\left(\frac{2\pi}{F}\boldsymbol{m}\times\boldsymbol{n}\right)J_{\boldsymbol{m}+\boldsymbol{n}}, \tag{20}$$

which generates the sine algebra or Fairlie–Fletcher–Zachos (FFZ) [8] algebra. This is exactly the Moyal bracket quantization of the area-preserving diffeomorphisms or symplectomorphism algebra on a $2-d$ torus. It should be mentioned that the deformation here (equation (20)) is the Moyal quantization, which is very different from the Drinfeld and Jimbo one [9], where the Hopf structure plays a crucial role.

Now let us give a construction of the quantum algebra $U_t(sl(2))$ in this scheme. Before continuing, we recall that this algebra is defined by the commutation relations [9]

$$\begin{aligned} &[X^+, X^-] = \frac{t^{2h} - t^{-2h}}{t - t^{-1}}, \\ &[h, X^\pm] = \pm X^\pm, \end{aligned} \tag{21}$$

where t is the deformed parameter, $t \neq 0, 1$. In the limit where $t \to 1$, the above equations reduce to ones defining the Lie algebra $sl(2)$.

The generators of $U_t(sl(2))$ can be embedded as follows [7]:

$$\begin{aligned} &X^+ = \frac{J_m - J_n}{t - t^{-1}}, \qquad X^- = \frac{J_{-m} - J_{-n}}{t - t^{-1}}, \\ &t^{+2h} = J_{m-n}, \qquad t^{-2h} = J_{n-m}. \end{aligned} \tag{22}$$

They satisfy the commutation relations given by equation (21) if t is given by

$$t = q^{m\times n/2}. \tag{23}$$

This concludes our brief review of the q-pseudo-differential operator and its related algebras. Now let us address our main goal, which will be expanded in the following sections.

3. Supersymmetric extension

In this section we start our generalization of the above results. We are looking for a supersymmetric extension of the sine algebra and the quantum algebra $U_t(sl(2))$. To do this let us begin by the realization of the supersymmetric sine algebra. This task is the subject of the following subsection.

3.1. Supersymmetric sine algebra

Our goal here is to extend the sine algebra to the supersymmetric case. To do this, let us introduce a fermionic algebra. We begin by considering the following matrices [10]:

$$f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{24}$$

It is easy to show that the generators f and f^+ are expanding a fermionic algebra. The latter is characterized by the relations

$$ff^+ + f^+f = 1, \qquad f^2 = 0 = (f^+)^2. \tag{25}$$

To obtain the supersymmetric extension of the operators J_m, we can proceed as follows:

$$\begin{aligned} &T_m = J_m \otimes (ff^+ + f^+f), \\ &S_m = J_m \otimes (ff^+ - f^+f), \end{aligned} \tag{26}$$

which define the so-called graded q-pseudo-differential operators. These operators can be written as

$$T_m = J_m \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S_m = J_m \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{27}$$

Using the last equation, we prove that the operators T_m and S_m satisfy the following relations:

$$\begin{aligned} &[T_m, T_n]_- = (q^{(m\times n)/2} - q^{-(m\times n)/2})T_{m+n}, \\ &[T_m, S_n]_- = (q^{(m\times n)/2} - q^{-(m\times n)/2})S_{m+n}, \\ &[S_m, S_n]_+ = (q^{(m\times n)/2} + q^{-(m\times n)/2})T_{m+n}. \end{aligned} \tag{28}$$

Let us remember that our goal here is to obtain a generalization of the sine algebra given by equation (20). To do this, we consider a q Fth root of unity. In this case, we show that the relations are verified:

$$
\begin{aligned}
[T_{\boldsymbol{m}}, T_{\boldsymbol{n}}]_- &= 2\mathrm{i}\sin\left(\frac{2\pi}{F}(\boldsymbol{m}\times\boldsymbol{n})\right)T_{\boldsymbol{m}+\boldsymbol{n}},\\
[T_{\boldsymbol{m}}, S_{\boldsymbol{n}}]_- &= 2\mathrm{i}\sin\left(\frac{2\pi}{F}(\boldsymbol{m}\times\boldsymbol{n})\right)S_{\boldsymbol{m}+\boldsymbol{n}},\\
[S_{\boldsymbol{m}}, S_{\boldsymbol{n}}]_+ &= 2\cos\left(\frac{2\pi}{F}(\boldsymbol{m}\times\boldsymbol{n})\right)T_{\boldsymbol{m}+\boldsymbol{n}}.
\end{aligned}
\tag{29}
$$

This set of equations generate exactly the supersymmetric sine algebra [11]. In the next section we show how to obtain $U_t(sl(2/1))$.

3.2. Quantum superalgebra $U_t(sl(2/1))$

The task of the present section is to realize the quantum superalgebra $U_t(sl(2/1))$ based on the defined supersymmetry operators equation (26). We begin by recalling that the quantum superalgebra $U_t(sl(2/1))$ can be viewed as a t-deformation of the classical Lie superalgebra $sl(2/1)$ through the t-deformed relations between a set of generators, denoted by e_1, e_2, f_1, f_2, $k_1 = t^{h_1}$, $k_1^{-1} = t^{-h_1}$, $k_2 = t^{h_2}$ and $k_2^{-1} = t^{-h_2}$, so that [12]

$$
\begin{aligned}
&k_1k_2 = k_2k_1, \qquad k_ie_jk_i^{-1} = t^{a_{ij}}e_j, \qquad k_if_jk_i^{-1} = t^{-a_{ij}}f_j,\\
&e_1f_1 - f_1e_1 = \frac{k_1^2 - k_1^{-2}}{t - t^{-1}}, \qquad e_2f_2 + f_2e_2 = \frac{k_2^2 - k_2^{-2}}{t - t^{-1}},\\
&e_1f_2 - f_2e_1 = 0, \qquad e_2f_1 - f_1e_2 = 0, \qquad e_2^2 = 0 = f_2^2,\\
&e_1^2e_2 - (t + t^{-1})e_1e_2e_1 + e_2e_1^2 = 0, \qquad f_1^2f_2 - (t + t^{-1})f_1f_2f_1 + f_2f_1^2 = 0.
\end{aligned}
\tag{30}
$$

The last two relations are called the Serre relations. The matrix (a_{ij}) is the Cartan matrix of $sl(2/1)$, i.e.

$$
(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}.
\tag{31}
$$

$U_t(sl(2/1))$ is a quasi-triangular Hopf superalgebra endowed with the $\boldsymbol{Z}_2$-graded Hopf algebra structure:

$$
\begin{aligned}
&\Delta(k_i) = k_i\otimes k_i, \qquad \Delta(e_i) = e_i\otimes\mathbf{1} + e_i\otimes k_i, \qquad \Delta(f_i) = e_i\otimes k_i^{-1} + \mathbf{1}\otimes f_i,\\
&\epsilon(k_i) = \mathbf{1}, \qquad \epsilon(e_i) = \epsilon(f_i) = 0,\\
&S(e_i) = -k_i^{-1}e_i, \qquad S(e_i) = -f_ik_i \qquad S(k_i^{\pm 1}) = k_i^{\pm 1}, \qquad i = 1, 2.
\end{aligned}
\tag{32}
$$

The $\boldsymbol{Z}_2$-grading of the generators are $[e_2] = [f_2] = 1$, and zero otherwise. The multiplication rule for the tensor product is $\boldsymbol{Z}_2$-graded and is defined for the elements a, b, c, d of $U_t(sl(2/1))$ by

$$
(a\otimes b)(c\otimes d) = (-1)^{[b][c]}(ac\otimes bd).
\tag{33}
$$

With the help of the symmetry operators equation (26), it is possible to give the following construction for the generators e_1, e_2, f_1, f_2, k_1 and k_2:

$$e_1 = \frac{T_{(m_1,m_2)} + T_{(-m_1,m_2)}}{t - t^{-1}} \otimes \mathbf{1}, \qquad f_1 = -\mathrm{i}\frac{T_{(m_1,-m_2)} - T_{(-m_1,-m_2)}}{t - t^{-1}} \otimes \mathbf{1},$$
$$k_1 = -\mathrm{i}T_{(-2m_1,0)} \otimes \mathbf{1}, \qquad k_1^{-1} = \mathrm{i}T_{(2m_1,0)} \otimes \mathbf{1},$$
$$k_2 = -\mathrm{i}T_{(-2m_1,0)} \otimes \begin{pmatrix} t^{-2} & 0 \\ 0 & t^2 \end{pmatrix}, \qquad k_2^{-1} = \mathrm{i}T_{(2m_1,0)} \otimes \begin{pmatrix} t^{2} & 0 \\ 0 & t^{-2} \end{pmatrix}, \tag{34}$$
$$e_2 = \frac{T_{(m_1,-m_2)} - T_{(-m_1,-m_2)}}{(t - t^{-1})^{\frac{1}{2}}} \otimes f, \qquad f_2 = \frac{T_{(m_1,m_2)} + T_{(-m_1,m_2)}}{(t - t^{-1})^{\frac{1}{2}}} \otimes f^{+}.$$

It turns out that the above generators satisfy the algebraic relations characterizing the quantum superalgebra $U_t(sl(2/1))$ as shown by equation (30), where the t-deformed parameter is given by

$$t = \mathrm{e}^{m_1 m_2}. \tag{35}$$

This is a way to prove that we can realize the $U_t(sl(2/1))$ by using the supersymmetry operators stated above.

4. Conclusion

In this paper, we have shown how the graded q-pseudo-differential operators can be used to obtain a supersymmetric extension of the sine algebra and the quantum algebra $U_t(sl(2))$. Also, we have realized the supersymmetric sine algebra and the quantum superalgebra $U_t(sl(2/1))$ with the help of q-operators graded with a fermionic algebra.

Acknowledgments

A J wishes to thank Professor S Randjbar-Daemi, Head of High Energy Section of the Abdus Salam International Centre for Theoretical Physics (AS-ICTP) for the kind hospitality of his section. He would also like to express his gratitude to Professor E H Saidi for his encouragement. The authors are grateful to Professor G Thompson for reading their manuscript and to Professor C K Zachos for drawing their attention to [11]. Finally, they would like to thank the referees for their useful suggestions and remarks.

References

[1] Gel'fand I M and Dickey L A 1976 *Funct. Ann. App.* **10** 252
Kac V G *et al Preprint* MIT/27/87
Radul A O 1991 *Phys. Lett.* B **265** 86
Kassel C 1992 *Commun. Math. Phys.* **140** 343
Khesin B, Lyubashenko V and Roger C 1994 *Preprint* hep-th/9403189
[2] Dereli T and Verçin A 1993 *J. Phys. A: Math. Gen.* **26** 6961
Dereli T and Verçin A 1992 *Phys. Lett.* B **288** 109
[3] Gould M D, Hibberd K E and Links J R 1996 *Phys. Lett.* A **212** 156
[4] Jellal A 2001 *Rep. Math. Phys.* **47** 1
[5] Kassel C 1992 *Commun. Math. Phys.* **140** 343
[6] Manin Yu 1989 *Commun. Math. Phys.* **123** 163
[7] El Kinani E H 1999 *Mod. Phys. Lett.* A **14** 1609
[8] Fairlie D B, Flecher P and Zachos C 1990 *J. Math. Phys.* **31** 1088
[9] Drinfeld V G 1986 *Proc. Int. Congr. Math.* ed A M Gleason (Berkeley: MSRI) p 798
Jimbo M 1985 *Lett. Math. Phys.* **10** 63
Jimbo M 1986 *Lett. Math. Phys.* **11** 247
[10] Jellal A 1999 *Mod. Phys. Lett.* A **14** 2253
[11] Fairlie D B and Zachos C K 1989 *Phys. Lett.* B **224** 101
[12] Chaichian M and Kulish P P 1990 *Phys. Lett.* B **234** 72
Abdsselam B, Arnaudon D and Bauer M 1997 *J. Phys. A: Math. Gen.* **30** 867